TEES

TEXAS ENGINEERING EXPERIMENT STATION
TEXAS A&M UNIVERSITY
COLLEGE STATION TEXAS 77843

Space, energy, matter and man, symbolize the broad areas into which the diverse TEES divisions conduct research and development

To disseminate knowledge is to disseminate prosperity — I mean general prosperity and not individual riches — and with prosperity disappears the greater part of the evil which is our heritage from darker times.

—Alfred Nobel

PROGRESS REPORT

to the

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Research Grant No. NGR-44-001-031

December 1, 1966

Report prepared by

Dr. Thomas J. Kozik
Principal Investigator

Submitted
by the
TEXAS ENGINEERING EXPERIMENT STATION
SPACE TECHNOLOGY DIVISION

TEXAS A&M UNIVERSITY
College Station, Texas

I. Introduction

The reinforced shell has become one of the principal structural members in those applications where the maximum in high strength and low weight is desired. However, the introduction of the stiffeners, whether they be of the rib type or laminates, introduces additional complexities into the already complex field of classical thin shell analysis.

The ability to predict the effects of the shell reinforcement has become one of the vital problems in evaluating shell strength characteristics. There is no known exact method of analyzing the reinforcements. However in general, the various assumptions and approximations made in regard to the reinforcement and its action on the shell lead to a solvable system of equations whose solutions predict the behavior of the shell within tolerable limits.

Normally, in the analysis of a reinforced shell, the specific configuration and mechanism of the reinforcement is taken into account during the development of the shell equations. As a consequence, the resulting analysis and equations are appropriate to only those types of reinforcements which satisfy the assumptions made during the development of the equations. Since most shell equations are programmed for use on a computer, the resulting program for the reinforced shell is also subject to the same restrictions as the analysis.

Because a reinforced shell usually exhibits directional strength properties and hence behaves in a manner similar to an unreinforced orthotropic shell, the former structure is termed a structurally orthotropic shell while the latter an intrinsically orthotropic shell. The similarity in the behavior of the strength characteristics suggests that the analysis of the reinforced shell may be approximated by the analysis of an intrinsically orthotropic shell possessing the same geometry and external loading.

The advantage in analyzing the intrinsically orthotropic shell lies in the fact that this shell does not possess the discontinuities that are introduced by stiffeners. Thus the techniques and assumptions employed in classical isotropic shell analysis may be utilized in deriving the orthotropic shell equations. More importantly, the effects of the stiffeners now manifest themselves solely in the orthotropic elastic constants. Since these quantities in turn appear as constant coefficients in the shell differential equations, it is obvious that the intrinsically orthotropic shell and hence the equivalent structurally orthotropic shell may be analyzed and the equations programmed without ever stating the value of the elastic constants or, equivalently, the type of stiffeners involved. Thus one single program, that for the intrinsically orthotropic shell, can account for a large class of stiffened shells.

In allowing an intrinsically orthotropic shell to represent a stiffened shell, the problem of a detailed analysis of the stiffeners is circumvented in the derivation of the shell equations. However, ultimately there does arise the problem of relating the properties of the stiffeners to the orthotropic elastic constants so that the intrinsically and structurally orthotropic shells will behave in a similar manner. This relation is termed a compliance of the elastic constants to the shell reinforcements and the rational means of its determination and solution are the primary concern of the research grant.

II. Procedure

The elastic constant compliance analysis of a structurally orthotropic shell requires three major areas of research. These areas are given as:

1. Conventional analysis of reinforced shells
2. Analysis of intrinsically anisotropic shells
3. Analysis of the compliance between anisotropic elastic constants and reinforcements

1. Conventional Analysis of Reinforced Shells

The first area of research analyzes the various conventional methods of dealing with reinforced shells. Such methods consist of a detailed analysis of the stiffeners and their effects on the shell differential equations. The importance of this area in regard to the present research effort is twofold.

First, it provides experimentally verifiable numerical values of stresses and displacements for various shell and stiffener configurations. Thus corresponding results found for similar shell and stiffener configurations by means of the compliance method can be checked for accuracy.

Secondly, this area of research provides the basis for the various assumptions made in regard to the stiffeners and their contribution to the shell strength characteristics. It has never been suggested that the conventional methods of analyzing reinforced shells yield results that will be inferior in accuracy to those found by the compliance method. Hence, many of the simplifying assumptions and approximations found in conventional shell stiffener analysis will be utilized in the compliance method of analysis.

2. Analysis of Intrinsically Anisotropic Shells

The second area of research deals with the analysis of intrinsically anisotropic shells. The term anisotropy rather than orthotropy is being utilized in that the research efforts cover not only orthotropic but also plane anisotropic elastic properties. Since the compliance method is based on the substitution of an intrinsically anisotropic unstiffened shell for a reinforced one, this area of investigation is responsible for the development of the shell differential equations.

The degree to which the behavior of the reinforced shell is approximated depends greatly on the choice of the intrinsically anisotropic shell model. Now first order shell theory, wherein the direct effects the transverse shear strains and stresses may be neglected, usually suffices for the conventional analysis of reinforced shells. In dealing with the intrinsically anisotropic model, the coefficients of the differential terms occurring in the shell differential equations are involved with the elastic constants. Because of the anisotropy, the magnitudes of these constant terms differ from their corresponding values in the isotropic case. Hence differential equation terms which normally may be neglected in isotropic first order shell theory may be significant in their contributions in anisotropic first order theory.

In order to determine the effects of anisotropy on first order shell theory, three separate investigations of the intrinsically anisotropic shell equations are presented. These investigations are as follows:

a. Conventional first order analysis where the effects of transverse shear stresses, strains and normal stresses normal to the surface are neglected

b. Second order analysis wherein the effects of transverse shear are explicitly incorporated into the shell equations

c. Elasticity analysis wherein all effects due to continuous media deformation are incorporated into the shell equations.

Each of the investigations attempts to answer specific questions regarding the role of anisotropy in shell theory.

The first investigation determines the form of the conventional first order anisotropic shell equations. Thus the effects of directional dependence on the differential terms occurring in the shell equations are discerned. This investigation also clarifies which, if any, additional differential terms arise over those found in isotropic first order theory.

The second investigation determines the explicit effects of transverse shear on the shell equations. In isotropic analysis, it is shown that consideration of these effects leads to the inclusion of differential terms which are small in comparison with the remaining terms of the differential equations. However, when dealing with the anisotropic shell, the interplay of the elastic constants may cause these very same terms to be larger and hence significant. In particular, the second order transverse shear terms of first order isotropic shell theory may become first order terms of anisotropic shell theory.

The third investigation is used primarily to determine boundary effects in the vicinity of edge shear loads. Since the elasticity formulation is less restrictive in assumptions than either first or second order shell theory, the results of this area of investigation also serve as a check on the results of the first two areas.

The investigation of intrinsically anisotropic shells covers not only orthotropy but also plane anisotropy of the elastic constants. One advantage in utilizing the latter condition is that it represents the more general of the two cases. Further, plane anisotropy allows a coupling to exist between the normal strains and shear stresses, a condition which is not present in orthotropy or isotropy.

Although a compliance will be sought between orthotropic elastic constants and stiffeners, there is no guarantee that all stiffeners can be represented by equivalent orthotropy. In fact, non-symmetrical stiffeners will result in a coupling effect between the extension and twist of a shell. Such a shell, therefore, behaves as one composed of a plane anisotropic material wherein coupling exists between the normal and shear strains. Hence, in order to account for such stiffener configurations, the plane anisotropic analysis for shells is investigated. Further, there still exists some question as to whether a symmetrically stiffened shell should not be treated as plane anisotropic, especially when such a shell is subjected to non-symmetrical loading.

3. Analysis of the Compliance Between Anisotropic Elastic Constants and Reinforcements

The third area of research is concerned with compliance studies and hence is responsible for the determination of the relations between the elastic constants and stiffeners of the intrinsically and structurally anisotropic shells. In order to effect a compliance, some criterion must be stated for the equality of the two shells. Although any criterion adopted must yield approximately the same results as an alternate criterion, the fact is that some criteria are more amenable to analytic treatment than others. The three criteria that are investigated are as follows:

a. Equality of stress resultants

b. Equality of equation coefficients

c. Equality of strain energy.

Each of the three criteria possess advantages. Thus, the equality of stress resultants for the intrinsically and structurally anisotropic shells is the simplest to apply but leads to severe problems when the shell is not symmetrically loaded. The equality of equation coefficients is the most accurate in that it ensures equality of the differential equations for the two shells and hence equality for the two solutions. The equality of strain energies possesses the greatest analytical advantages when the shell is not symmetrically loaded. It also is especially advantageous in resolving the stiffener effects for bending and membrane stresses.

III. Results

The results to be reported cover the three major areas of research described in Section II. A considerable portion of the work is incorporated in doctoral dissertations and masters theses. Rather than reproduce the details found in these works, only the conclusions and salient features of the analyses will be given, the assumption being that if more detailed information is desired, the appropriate work can be obtained from Texas A&M University. Further, many of the theses and dissertations are being abstracted for publication and hence a second source of these works should be available.

A number of separate investigations have been made within the major areas of research. A summary of these investigations and their appropriate area of research are given as follows:

1. Conventional Analysis of Reinforced Shells
   a. Literature survey and evaluation of stiffened plate analysis
   b. Analysis of meridionally and longitudinally rib stiffened shells of revolution subjected to axially symmetric loadings.
2. Analysis of Intrinsically Anisotropic Shells
   a. Development of conventional first order orthotropic displacement equations for shallow and non-shallow shells.
   b. Determination of the effects of transverse shear on the orthotropic shell equations
   c. Elasticity solutions for isotropic and orthotropic shells of revolution subjected to symmetrical loading.

d. Analysis of symmetrically loaded orthotropic shells of revolution by means of the matrix-displacement method.

e. Analysis of Sanders' type displacement equations for plane anisotropic shells of revolution.

3. Analysis of the Compliance Between Anisotropic Elastic Constants and Reinforcements

a. General compliances by means of equality of stress resultants

b. Compliance analysis for a rib stiffened shallow cylindrical panel by means of an energy criterion.

A detailed discussion of the various topics follows:

1a. Literature Survey and Evaluation of Stiffened Plate Analysis

The stiffened plate has been used more extensively as a structural member than the stiffened shell. As a consequence, a body of literature has accumulated dealing with the detailed analysis of plate stiffeners. Since a plate is a specialized case of a shell and since many of the principles elucidated in the analysis of reinforced plates are directly applicable to reinforced shell analysis, it was believed that a survey study of stiffened plate analysis would be advantageous to the project. Such a study has been made and used as the basis of a Master of Science thesis (1)[1].

One of the principal results of the plate survey was to note that the use of an intrinsically orthotropic plate to predict the behavior of a structurally orthotropic one had been investigated in some detail. The

---

[1] Numbers in parenthesis refer to references given at the end of the report.

works of Dow (2) Witt (3) and Hoppmann (4) demonstrated analytically and experimentally that the intrinsically orthotropic plate with suitably defined elastic constants was an acceptable model for the corresponding reinforced plate. Huffington (5), (6), (7) further demonstrated that it was possible to derive a general compliance formula for the elastic constants which would account for the majority of rib-stiffened plates.

1b Structurally Orthotropic Shells of Revolution

The matrix displacement method is used to analyze structurally orthotropic shells of revolution under both axisymmetric and asymmetric loading (8). Both meridional and circumferential rib stiffeners are considered in the analysis. In carrying out the solution the techniques described under "Intrinsically Orthotropic Shells of Revolution," are used. The only change necessary is in the calculation of the strain energy. The strain energy values of the ribs are simply added to the strain energy of the unstiffened shell.

The stiffeners are assumed to be closely and uniformly distributed over the element so that the contribution to the strain energy from the stiffeners may be computed on the basis of beam properties per unit length. The centroidal axes of the stiffeners are not required to lie on the middle surface of the shell. This eccentricity is incorporated by expressing the strain displacement relationships of the stiffeners as functions of the middle surface of the shell. It is assumed that plane sections remain plane. Numerical examples were compared to known solutions. Good agreement was obtained in all cases.

2a. First Order Shallow and Cylindrical Shell Equations

Conventional small displacement linear elastic shell theory is referred to as first order shell analysis. The bases of this analysis are the Kirchoff hypotheses which are given as follows:

(i) Line segments initially normal to the shell middle surface remain so after deformation.

(ii) Normal stresses normal to the shell middle surface may be neglected in comparison with the in-plane stresses.

(iii) A line segment normal to the shell middle surface suffers neither extension nor contraction.

Various isotropic formulations of the shell equations are given on the basis of the above assumptions. Though the formulations may differ from each other, the differences in either the structure of the equations or the numerical accuracy does not exceed an order of magnitude equal to ($k\delta$). However, the inherent error in the use of the Kirchoff hypotheses is at least of order ($k\delta$) and hence the variations in the various formulations ceases to be of importance. In fact, first order shell theory may be characterized by the fact that all terms of order ($k\delta$) in comparison to unity may be neglected.

In dealing with an orthotropic material, there existed some question as to whether the interplay of material constants would not alter the structure of the shell equations. In order to answer the question, a first order orthotropic formulation of the shell equations was derived. The resulting equations were presented in displacement rather than stress resultant form and further, were simplified so as to be applicable to a shell wherein the transverse shear stress resultants could be neglected in the first two equilibrium equations. Thus the equations were applicable to right circular cylindrical shells and to shallow shells.

The principal results of the investigation were two in number. First, the structure of the orthotropic equations was the same as for the corresponding isotropic ones. Second, a set of displacement equations for an orthotropic material were made available and to which ultimately compliance results would be applied in order to determine deflections for stiffened cylindrical and shallow shells.

The resulting equations and notations are given as follows:

a. Hooke's Law for an orthotropic material

For an orthotropic material, the generalized Hooke's Law may be written as the following:

$$
\begin{aligned}
e_{\alpha\alpha} &= a_{11}\sigma_{\alpha\alpha} + a_{12}\sigma_{\beta\beta} + a_{13}\sigma_{\gamma\gamma} \\
e_{\beta\beta} &= a_{21}\sigma_{\alpha\alpha} + a_{22}\sigma_{\beta\beta} + a_{23}\sigma_{\gamma\gamma} \\
e_{\gamma\gamma} &= a_{31}\sigma_{\alpha\alpha} + a_{32}\sigma_{\beta\beta} + a_{33}\sigma_{\gamma\gamma} \\
e_{\beta\gamma} &= a_{44}\sigma_{\beta\gamma} \\
e_{\alpha\gamma} &= a_{55}\sigma_{\alpha\gamma} \\
e_{\alpha\beta} &= a_{66}\sigma_{\alpha\beta}
\end{aligned}
$$

The quantities "a_{ij}" are the elastic constants of the material and possess diagonal symmetry, that is

$$a_{ij} = a_{ji} \quad (i \neq j)$$

b. Notation

α, β Principal curvilinear coordinates of shell middle surface

A, B	Lame surface parameters of the middle surface
δ	Shell thickness
k_α, k_β	Principal curvatures of the surface
u, v, w	Components of the displacement vector of the middle surface
$P_\alpha, P_\beta, P_\gamma$	Components of external load per unit middle surface area
K_α, K_β	Curvature changes of the middle surface
H	Mean surface curvature $\left(\frac{k_\alpha + k_\beta}{2}\right)$
K	Gaussian curvature

c. Displacement Equations

$$\frac{a_{22}}{(a_{11}a_{22}-a_{12}^2)} B \frac{\partial}{\partial\alpha}\left\{\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(Bu) + \frac{\partial}{\partial\beta}(Av)\right]\right\}$$

$$+ \frac{(a_{22}+a_{12})}{(a_{11}a_{22}-a_{12}^2)} ABKu + \frac{1}{a_{66}} A\frac{\partial}{\partial\beta}\left\{\frac{1}{AB}\left[\frac{\partial(Au)}{\partial\beta} - \frac{\partial}{\partial\alpha}(Bv)\right]\right\}$$

$$+ \frac{2a_{22}}{(a_{11}a_{22}-a_{12}^2)} B\frac{\partial(Hw)}{\partial\alpha} - \frac{(a_{22}+a_{12})}{(a_{11}a_{22}-a_{12}^2)} B k_\beta \frac{\partial w}{\partial\alpha}$$

$$+ \frac{(a_{22}-a_{11})}{(a_{11}a_{22}-a_{12}^2)} \frac{1}{AB}\left(\frac{\partial B}{\partial\alpha}\right)^2 u + \left[\frac{(a_{22}+a_{12})}{(a_{11}a_{22}-a_{12}^2)} - \frac{2}{a_{66}}\right]\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right)u$$

$$- \left[\frac{(a_{22}-a_{12})}{(a_{11}a_{22}-a_{12}^2)} - \frac{2}{a_{66}}\right]\frac{\partial^2 v}{\partial\alpha\partial\beta} + \frac{(a_{22}+a_{11})}{(a_{11}a_{22}-a_{12}^2)}\frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial v}{\partial\beta}$$

$$+ \left[\frac{(a_{22}+a_{12})}{(a_{11}a_{22}-a_{12}^2)} - \frac{2}{a_{66}}\right]\frac{1}{AB}\frac{\partial B}{\partial\alpha}\frac{\partial A}{\partial\beta} v \quad +$$

$$+\frac{(a_{22}-a_{11})}{(a_{11}a_{22}-a_{12}^2)}\,k_\beta\frac{\partial B}{\partial\alpha}\,w \;+\; \frac{AB}{\delta}\,P_\alpha = 0$$

$$\frac{a_{11}}{(a_{11}a_{22}-a_{12}^2)}\,A\frac{\partial}{\partial\beta}\left\{\frac{1}{AB}\left[\frac{\partial(Av)}{\partial\alpha}+\frac{\partial(Bu)}{\partial\beta}\right]\right\}$$

$$+\frac{(a_{11}+a_{12})}{(a_{11}a_{22}-a_{12}^2)}\,ABKv \;+\; \frac{1}{a_{66}}\,B\frac{\partial}{\partial\alpha}\left\{\frac{1}{AB}\left[\frac{\partial(Bv)}{\partial\alpha}-\frac{\partial(Au)}{\partial\beta}\right]\right\}$$

$$+\frac{2a_{11}}{(a_{11}a_{22}-a_{12}^2)}\,\frac{A\,\partial(Hw)}{\partial\beta} \;-\; (a_{11}+a_{12})\,A\,k_\alpha\frac{\partial w}{\partial\beta}$$

$$+\frac{(a_{11}-a_{22})}{(a_{11}a_{22}-a_{12}^2)}\,\frac{1}{AB}\left(\frac{\partial A}{\partial\beta}\right)^2 v + \left[\frac{(a_{11}+a_{12})}{(a_{11}a_{22}-a_{12}^2)}-\frac{2}{a_{66}}\right]\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right)v$$

$$+\left[\frac{(a_{11}+a_{12})}{(a_{11}a_{22}-a_{12}^2)}-\frac{2}{a_{66}}\right]\frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{\partial B}{\partial\alpha}\,u \;+\; \frac{(a_{11}+a_{22})}{(a_{11}a_{22}-a_{12}^2)}\,\frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial u}{\partial\alpha}$$

$$-\left[\frac{(a_{11}-a_{12})}{(a_{11}a_{22}-a_{12}^2)}-\frac{2}{a_{66}}\right]\frac{\partial^2 u}{\partial\alpha\partial\beta} \;+\; \frac{(a_{11}-a_{22})}{(a_{11}a_{22}-a_{12}^2)}\,k_\alpha\frac{\partial A}{\partial\beta}\,w$$

$$+\;\frac{AB}{\delta}\,P_\beta = 0$$

$$(a_{11}+a_{12})\frac{\partial}{\partial\alpha}(Bk_\beta u) + (a_{22}+a_{12})\frac{\partial}{\partial\beta}(Ak_\alpha v) \;-$$

$$-2H\left[a_{11}\frac{\partial}{\partial\alpha}(Bu) + a_{22}\frac{\partial}{\partial\beta}(Av)\right] + (a_{11}-a_{12})\,B\,k_\alpha\frac{\partial u}{\partial\alpha}$$

$$+2AB\left\{\left[\frac{(a_{11}+a_{22})}{2} + a_{12}\right]K - (a_{11}+a_{22})H^2\right\}w + \frac{(a_{11}-a_{22})}{2}k_\beta^2 ABw$$

$$-\frac{(a_{22}-a_{11})}{2}k_\gamma^2 ABw + \frac{\delta^2}{12}\left\{a_{22}\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\nabla_e^2 w\right) + a_{11}\frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\nabla_e^2 w\right)\right.$$

$$+\left[\frac{2}{a_{66}}(a_{11}a_{22}-a_{12}^2) - (a_{22}+a_{12})\right]\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial K_\beta}{\partial\alpha}\right)$$

$$\left.+\left[\frac{2}{a_{66}}(a_{11}a_{22}-a_{12}^2) - (a_{11}+a_{22})\right]\frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial K_\alpha}{\partial\beta}\right)\right\} + \frac{(a_{11}a_{22}-a_{12}^2)}{\delta}ABP_\gamma$$

$$= 0$$

In the above equation, the elliptic operator ∇_e^2 is defined as

$$\nabla_e^2 = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)\right]$$

and the curvature changes K_α, K_β as;

$$K_\alpha = -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta}$$

$$K_\beta = -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}$$

d. Constitutive Equations

The stress resultant deformation relations are given as:

$$T_{\alpha\alpha} = \frac{\delta}{(a_{11}a_{22}-a_{12}^2)}(a_{22}e_{\alpha\alpha} - a_{12}e_{\beta\beta}) \qquad M_{\alpha\alpha} = \frac{\delta^3}{12(a_{11}a_{22}-a_{12}^2)}(a_{22}K_{\alpha} - a_{12}K_{\beta})$$

$$T_{\alpha\beta} = \frac{2\delta}{a_{66}}e_{\alpha\beta} \qquad M_{\alpha\beta} = \frac{2\delta^3}{12a_{66}}\tau$$

$$T_{\beta\beta} = \frac{\delta}{(a_{11}a_{22}-a_{12}^2)}(a_{11}e_{\beta\beta} - a_{12}e_{\alpha\alpha}) \qquad M_{\beta\beta} = \frac{\delta^3}{12(a_{11}a_{22}-a_{12}^2)}(a_{11}K_{\beta} - a_{12}K_{\alpha})$$

2b. Determination of the Effects of Transverse Shear on the Orthotropic Shell Equations

The effect of transverse shear deformation on stresses and deflections in thin shells of revolution was investigated by considering several particular types of shells (9). First and second approximation solutions with and without transverse shear deformation were derived and a numerical solution was obtained for a semi-infinite circular cylindrical shell. First approximation solutions with and without transverse shear deformation were determined for spherical shells and indicated for ellipsoidal shells of revolution.

As a result of the above work, the following conclusions were made:

(i) From an engineering point of view, transverse shear deformations may be neglected in analyzing thin shells of revolution.

(ii) From an engineering point of view, second approximation theories appear to offer no advantages over first approximation theories for analyzing thin shells of revolution.

2c. Elasticity Solutions for Isotropic and Orthotropic Shells of Revolution Subjected to Symmetrical Loading

A system of three-dimensional partial differential equations in terms of displacements referenced to curvilinear coordinates was developed to describe the behavior of an orthotropic shell with arbitrary loadings and boundary conditions (10). The system of equations represent a generalization of the isotropic Navier elasticity equations and are exact within the limitations of assumed small displacements and linear stress-strain relations.

The three-dimensional system was then reduced to the axisymmetric case which describes the behavior of a symmetrically loaded, isotropic shell of revolution with symmetrical boundary conditions. The resulting two-dimensional system of equations was numerically approximated by the method of divided differences and the resulting equations were solved by matrix inversion.

2d. Intrinsically Orthotropic Shells of Revolution

The matrix displacement method is used to analyze intrinsically orthotropic shells of revolution under both axisymmetric and asymmetric loading (8). The generic element used in the analysis is that of an axisymmetric shell segment in which the angle between the axis of revolution and the tangent to the meridian at any point on the segment is described as a quadratic in the meridional coordinate. The deformation of the shell can then be approximated by three linear displacement components and a rotation along each nodal* circle by using suitable displacement functions. The displacement functions were chosen such that the two conditions required by the matrix displacement method, which may be considered as an application of the Rayleigh-Ritz method to discrete elements within a structure, were satisfied. These conditions are

(i) The geometric boundary conditions between elements should be satisfied.

(ii) The displacement functions should be the lower order terms of a set of functions which is complete.

The two-dimensional problem created by the asymmetric loadings may be reduced essentially to a one-dimensional problem if the loads and deflections are expressed as a Fourier series in the circumferential direction. The displacement functions used are Fourier trigonometric series in the theta direction with coefficients which are polynomials in the meridional coordinate , S. The meridional displacement was a cubic function of S , and the tangential and normal displacements were linear functions of S.

*The term "node" is used to designate the edge of an element.

The equations of equilibrium are obtained from the application of the principle of minimum potential energy.

Since the internal energy and external potential energy are not coupled, each harmonic of loading can be handled independently and the final solution considered as the sum of partial solutions.

In order to perform the calculations necessary to obtain solutions, a computer program was written in FORTRAN IV for the IBM 7094. The program is composed of the main program, 13 subroutines, and one function subprogram. The purpose of the program is to produce, for each harmonic, the displacements, reactions, and stress resultants of nodal points given the loading, geometry, and nodal locations. The program was used to compare a large number of example problems with other solutions that are considered to be accurate. The comparisons were excellent.

2c. Analysis of Sander's Type Displacement Equations for Plane Anisotropic Shells of Revolution

When dealing with an orthotropic material, the resulting shell differential equations were of the same mathematical structure as had been encountered for an isotropic material. In order to determine the effects of anisotropy on first order shell theory and still have a relatively simple model, a plane anisotropic material has been hypothesized wherein the effects of in-plane shear affect the in-plane normal strains. For such a material, the generalized Hooke's Law was given as

$$e_{\alpha\alpha} = a_{11}\sigma_{\alpha\alpha} + a_{12}\sigma_{\beta\beta} + a_{13}\sigma_{\gamma\gamma} + a_{16}\sigma_{\alpha\beta}$$
$$e_{\beta\beta} = a_{21}\sigma_{\alpha\alpha} + a_{22}\sigma_{\beta\beta} + a_{23}\sigma_{\gamma\gamma} + a_{26}\sigma_{\alpha\beta}$$
$$e_{\gamma\gamma} = a_{31}\sigma_{\alpha\alpha} + a_{32}\sigma_{\beta\beta} + a_{33}\sigma_{\gamma\gamma} + a_{36}\sigma_{\alpha\beta}$$
$$e_{\beta\gamma} = a_{44}\sigma_{\beta\gamma} + a_{45}\sigma_{\alpha\gamma}$$
$$e_{\alpha\gamma} = a_{54}\sigma_{\beta\gamma} + a_{55}\sigma_{\alpha\gamma}$$
$$e_{\alpha\beta} = a_{61}\sigma_{\alpha\alpha} + a_{62}\sigma_{\beta\beta} + a_{63}\sigma_{\gamma\gamma} + a_{66}\sigma_{\alpha\beta}$$

where $a_{ij} = a_{ji}$ $(i \neq j)$.

Although the first order shell theory stress resultant equations are more frequently encountered than the corresponding displacement equations, the latter possess certain advantages over the former. To begin with, the number of equations are reduced to only three, secondly, the compatibility equations are identically satisfied, and thirdly, the boundary conditions are simple to state. However, the disadvantages of the displacement formulation are equally strong. Thus, the order of derivatives is usually higher than for the stress resultant formulation and most importantly, the simplification of the equations by means of the Kirchoff criteria is involved. This last disadvantage of the displacement equations is the chief reason for the displacement formulation being relegated to shallow and cylindrical shells.

The Kirchoff error criteria of $(k\xi)$ is derived on the basis of strain of stress resultant arguments. Hence in simplifying the first order shell equations, strains and stresses of order $(k\xi)$ in comparison to unity may be neglected without introducing additional errors in the shell equation formulations. Now strains may be shown to be related to displacements by means of differential operators. However, not all displacements are related to strains. As a shell is subjected to load, not only does it deform and thus acquire strain, but it also may rotate and translate and thus locally acquire rigid body motions. These latter quantities are also related to the displacements by means of differential operators. Thus, the resulting displacement of a point in a shell is composed of the effects due to the strain and the rigid body motions of that point.

The smallness of strain does not assure the smallness of rigid body motions. Depending on the end fixity and the geometry of the shell, the rigid body motions may be an order of magnitude larger than the strains. As a consequence, the displacement terms involved with rigid body motions also may be an order of magnitude larger than the corresponding displacement terms involved in the strain expressions.

In stating the shell equations in displacement form, there arises the problem of determining that portion of the displacement due to strains and the portion due to the rigid body motions. This separation is important if the Kirchoff error criteria is to be used in the simplification of the equations since the criteria is applicable only to that portion of the displacement due to strain. However, it is virtually impossible to form such a separation of the displacements in the conventional first order shell equations and hence two possible altererernatives result.

The first alternative is to simplify all displacement components by means of the Kirchoff error criteria. Such a procedure is tantamount to assuming that local rigid body motions are of the same order of magnitude as the strains, an assumption justified for shallow shells and for certain end restraints. The second alternative is to not simplify the displacement equations and thus allow terms of order $(k\delta)^2$ to exist. Though the latter alternative is the more general, the resulting equations become much more difficult to solve than the equations of the first alternative.

There exists yet a third alternative and one which has been adopted for the research reported in this section. This alternative is based on Sander's theory (11) wherein the first order shell equations are derived in a manner such that the displacement components are devoid of local rigid body motions. Thus, the resulting displacement equations may be simplified by the Kirchoff error criteria without introducing the restrictions or complications noted in the previous paragraph.

Sanders equations for first order shell theory as found in the literature were stated in stress resultant form. However, a displacement formulation was desired and hence a separate development had to be performed. The resulting displacement equations were then simplified so as to be applicable to bodies of revolution subjected to axially symmetrical loadings. However, unlike the isotropic or orthotropic case, the equations did not reduce to the conventional first order formulation. The primary cause for this nonconvergence was the shear coupling found between the normal and shear strains in a plane anisotropic material.

In order to have maintained symmetry in the shell problem under discussion, three conditions had to be satisfied. First, the loads had to be symmetrical, second, the geometry had to possess symmetry, and third the constitutive equations also had to possess symmetry of deformation. The inclusion of the shear coupling destroyed the symmetry of deformations on the constitutive equations and hence also destroyed the symmetry of the problem. The net result was the necessity of including the circumferential displacement component, a term not found in either the isotropic or orthotropic formulations of the same equations.

The details of the development of the Sander's type displacement equation for a plane anisotropic material are contained in a doctoral dissertation (12). In addition to the derivation of the equations, an analytical solution is given for a right circular cylindrical shell subjected to a symmetrical edge loading. Because of the plane anisotropy and the consequent loss of symmetry, the circumferential displacement terms are contained in the equations and these terms are directly responsible for a phenomena not previously noted and not present in either an isotropic or orthotropic material. That is, the form of solution for the right circular cylindrical shell is dependent upon the degree of plane anisotropy as judged by the magnitude of the elastic constants a_{16} and a_{26} in comparison to the other constants. Further, for certain ratios of the elastic constants, the shell evidences an instability in displacements independent of loading.

The last result is so striking that at the present time all that can be said is that this conclusion is logically derived from the equations. At the present time, a paper is being prepared for publication in order to have critical reviews of the work. If the reviews prove favorable, a limited testing program will be inaugurated to verify the results experimentally.

3a. <u>General Compliances by Means of Equality of Stress Resultants</u>

The stress strain relations for an intrinsically orthotropic material are given as;

$$
\begin{aligned}
e_{\alpha\alpha} &= a_{11}\sigma_{\alpha\alpha} + a_{12}\sigma_{\beta\beta} + a_{13}\sigma_{\gamma\gamma} \\
e_{\beta\beta} &= a_{21}\sigma_{\alpha\alpha} + a_{22}\sigma_{\beta\beta} + a_{23}\sigma_{\gamma\gamma} \\
e_{\gamma\gamma} &= a_{31}\sigma_{\alpha\alpha} + a_{32}\sigma_{\beta\beta} + a_{33}\sigma_{\gamma\gamma} \\
e_{\beta\gamma} &= a_{44}\sigma_{\beta\gamma} \\
e_{\alpha\gamma} &= a_{55}\sigma_{\alpha\gamma} \\
e_{\gamma\gamma} &= a_{66}\sigma_{\gamma\gamma}
\end{aligned}
$$

Diagonal symmetry in the elastic constants requires that $a_{ij} = a_{ji}$ ($i \neq j$). In applying the above equations to a shell, the α and β directions correspond to the principle directions on the surface while γ is measured normal to the surface.

In first order theory, the constitutive equations involve only six stress resultants. The relation between the transverse shear stress resultants and the corresponding strain is suppressed since first order theory negates the existence of transverse shear strains. Thus, the stress resultant deformation equations are given as;

$$T_{\alpha\alpha} = \frac{\delta}{(a_{11}a_{22}-a_{12}^2)}(a_{22}e_{\alpha\alpha}-a_{12}e_{\beta\beta}) \qquad M_{\alpha\alpha} = \frac{\delta^3}{12(a_{11}a_{22}-a_{12}^2)}(a_{22}K_{\alpha}-a_{12}K_{\beta})$$

$$T_{\beta\beta} = \frac{\delta}{(a_{11}a_{22}-a_{12}^2)}(a_{11}e_{\beta\beta}-a_{12}e_{\alpha\alpha}) \qquad M_{\beta\beta} = \frac{\delta^3}{12(a_{11}a_{22}-a_{12}^2)}(a_{11}K_{\beta}-a_{12}K_{\alpha})$$

$$T_{\alpha\beta} = \frac{2\delta}{a_{66}}e_{\alpha\beta} \qquad M_{\alpha\beta} = \frac{\delta^3}{6a_{66}}\tau$$

In these expressions, δ is the shell thickness, K_{α} and K_{β} the middle surface curvature changes in the α and β directions, respectively, and τ is the twist of the middle surface.

Since there are only four independent elastic constants but six independent stress resultants, then only four of the six stress resultants can be made to comply with the stress resultants of the stiffened shell. However, the choice of the four to be used is not completely arbitrary. Note that the constant a_{66} appears only in the membrane shear force and twisting moment. Thus, only two of these stress resultants can be made to comply with those of the corresponding stiffened shell.

If the shell reinforcement is of an attached type, then a stress resultant compliance based on first order shell theory may yield satisfactory results. However, the degree of satisfaction will depend greatly on the inefficiency of the attaching medium in making the reinforcement an integral part of the shell. As an example, consider a rib stiffened shell where the ribs are attached to the shell by means of rivets or some other similar device. If the rib attaching device is such as to allow some relative motion between the rib and the shell in a direction tangent to the shell surface, then the effects of the rib membrane forces on the shell stress condition will be small and hence the ribs will be used to resist only bending and twisting moments. Under these circumstances, a stress resultant compliance may be found by equating the bending and twisting moment stress resultants of the intrinsically and structurally orthotropic shells.

3b. Compliance Analysis for a Rib-Stiffened Shallow Cylindrical Panel by Means of an Energy Criteria

In the literature survey of stiffened plate analysis, it had been noted that Huffington (5) (6) (7) succeeded in analyzing a rib stiffened plate by compliance means using an energy criteria. Furthermore, Huffington's work contained equations relating the orthotropic bending rigidities to the stiffener effects. The equations were stated in terms of the two dimensional stress function and hence were applicable to any rib stiffener configuration.

The shallow shell represents the simplest transition from a plate to a shell. As a consequence, it is relatively simple to identify those terms which arise because of the curvature of the shell and hence do not appear in the plate formulation. Thus in comparing the shallow shell displacement equations with the corresponding plate equation, it is found that the former contain tangential middle surface displacement components not found in the latter. Further, the conditions of equilibrium in directions tangent to the middle surface are not identically satisfied for the shell as they are in the plate. The net result is that the shallow shell displacement formulation contains three equations while the plate formulation contains but one equation. The additional two equations of the shell formulation represent the curvature manifestation of in-plane or membrane forces which together with the bending rigidities of the shell resist external loading.

The bending rigidities of a shallow shell are not affected by the curvatures. As noted in the previous paragraph, the curvatures primarily manifest themselves through the inclusion of membrane forces. If the shell is sufficiently shallow, the bending rigidities of a reinforced or intrinsically orthotropic shell segment should differ but by a small amount from the corresponding bending rigidities of a plate. Hence there should exist a range of values of shell shallowness for which Huffington's plate compliance equations will be directly applicable to the shell.

The purpose of the research work reported in this section is to determine the applicability of Huffington's plate compliance equations to shallow shell configurations. A rib-stiffened shallow cylindrical panel as shown in Fig. 1 is subjected to a uniform pressure and to simply supported boundary conditions. The dimensions of the panel and the rib reinforcements are held fixed but the shallowness of the shell is varied by varying the radius of the panel. The above panel is solved for the maximum normal deflection using Huffingtons compliance equations for the bending rigidities associated with an intrinsically orthotropic shallow cylindrical panel of the same geometric configuration, the same boundary conditions and subjected to the same external load as the stiffened panel. In order to gain some idea of the accuracy of the results, solutions obtained from a set of equations presented by Flugge for the same rib-stiffened problem are compared to the results obtained by the compliance analysis.

Huffington's work is concerned with plate analysis. In the absence of in-plane external or boundary loads, the load carrying capacity of the plate is solely determined by the bending rigidities. Thus Huffington's compliance equations, which are applicable only to the bending rigidities, completely define compliance analysis for plates. In a shallow shell, terms additional to those found in the plate equations arise because of the presence of membrane forces. Unlike the plate, these forces exist whether or not the external loading is or is not normal to the shell surface. Hence, the membrane forces and their effect form an integral part of shell analysis.

In the orthotropic shallow shell equations, groupings of orthotropic elastic constants prefix the various differential terms. Certain of the groupings are identified as the shell bending rigidities and the compliance of these groups to the shell stiffeners is accomplished by Huffingtons equations. However, other groupings of the elastic constants prefix terms which can be directly related to the membrane effects. As such, for convenience of expression, these groupings may be termed membrane rigidities. Since plate analysis does not contain the membrane effects, then there does not exist any compliance equation derived from plate analysis which relate the membrane rigidities to the stiffeners.

The solution of the rib-stiffened cylindrical panel by compliance means necessitated the determination of the membrane rigidities. Three possible expressions for the compliance of the membrane rigidity terms to the stiffeners were investigated. These expressions were based on three possible principles given as follows:

(i) Consistent Definition

The membrane stiffeners coefficients are groupings of elastic constants and the bending stiffeners are also groupings of the same elastic constants. Hence, as might be expected, knowing one set of stiffeners, the other set may be derived. ~~In fact, the two sets of~~ stiffeners are proportional to each other, the constant of proportionality being $\delta^2/12$, where δ is the shell thickness. Letting D_{ij} be the stiffeners associated with bending and C_{ij} the stiffeners associated with bending. Then

$$D_{ij} = \frac{\delta^2}{12} C_{ij}$$

(ii) Increased Area Effect

The principle behind the second possibility is that the stiffeners increase the area of the shell resisting membrane forces. As a consequence, this increase in area should cause a corresponding decrease in the membrane forces. Hence, the compliances of the membrane rigidities should be based solely on the effect the stiffeners have on the cross sectional area of the shell.

This method of dealing with the membrane stiffener effects is used by Flügge in deriving his rib-stiffened cylindrical panel equations. So far as the present research effort is concerned, it has the disadvantage in that it separates bending and membrane rigidities into the separate and non-connected areas. It also destroys the identity of the individual elastic constants which compose the two rigidities.

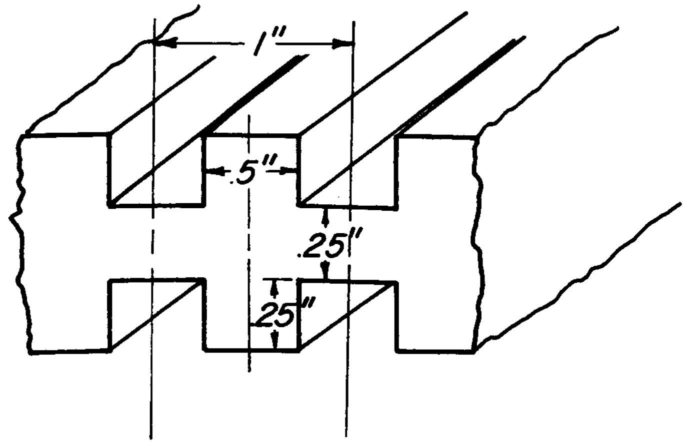
(iii) Isotropic Effect

The third possibility is that the stiffeners have no effect on the membrane rigidities. Hence, the shell behaves as an isotropic unstiffened member so far as membrane effects are concerned. For extremely shallow shells, such an an assumption is justified in that the membrane rigidities are insignificant in affecting the load carrying capacity of the structure. However, when the shallowness of the shell decreases and the structure becomes identifiable as a curved panel, then the neglect of the stiffener on the membrane rigidity becomes questionable.

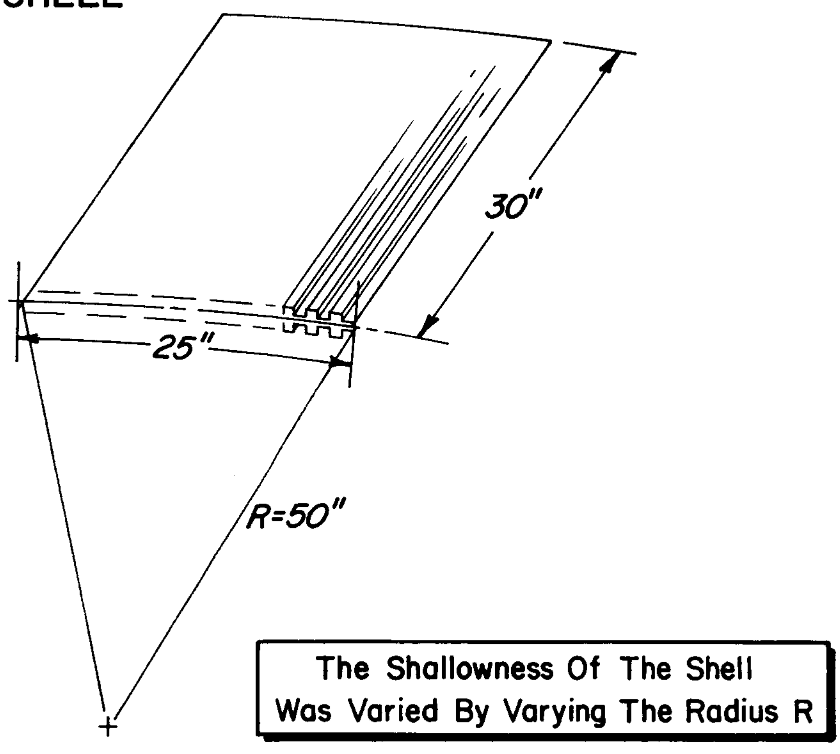
Each of the three conditions had been used in the shallow shell compliance equations. The results are depicted in Fig. 2 and the details of the analysis are contained in a Master of Science thesis (13). Because the bending and membrane rigidities would be varied, each of the curves are identified by means of three letters. The first letter indicates the formulation being utilized. Thus V stands for the displacement equations first derived by Vlasov for an isotropic media and then extended by Kozik to an orthotropic material. These equations are given in the report in section 2a. The F stands for the equations given by Flügge for a rib stiffened panel. The second letter stands for the membrane rigidities being used. Thus I stands for the isotropic rigidities, that is, not taking into account the stiffener effects on the membrane forces. The F stands for the membrane rigidities given by Flügge and the O stands for the membrane rigidities that are derived from Huffington's bending rigidities. The third letter indicates the manner in which the bending rigidities had been calculated. The H, I and F follow the same code as used for the membrane rigidities.

The results indicate that there does exist a range of shell shallowness wherein Huffington's plate compliance equations are appropriate for shell compliances. The results also indicate the pronounced effect the bending rigidities have in decreasing the maximum deflection from the value found in the isotropic case. If Flügge's results are accepted as being reasonable approximations to the three deflections experienced in the reinforced panel, then the curves also indicate that the intrinsically orthotropic panel equations with the bending rigidities as given by Huffington and isotropic membrane rigidities closely approximate the deflection of the rib stiffened panel. Note that the use of orthotropic membrane rigidities results in significantly lower values of deflection than experienced from Flügge's analysis.

REPEATING SECTION OF SHALLOW SHELL

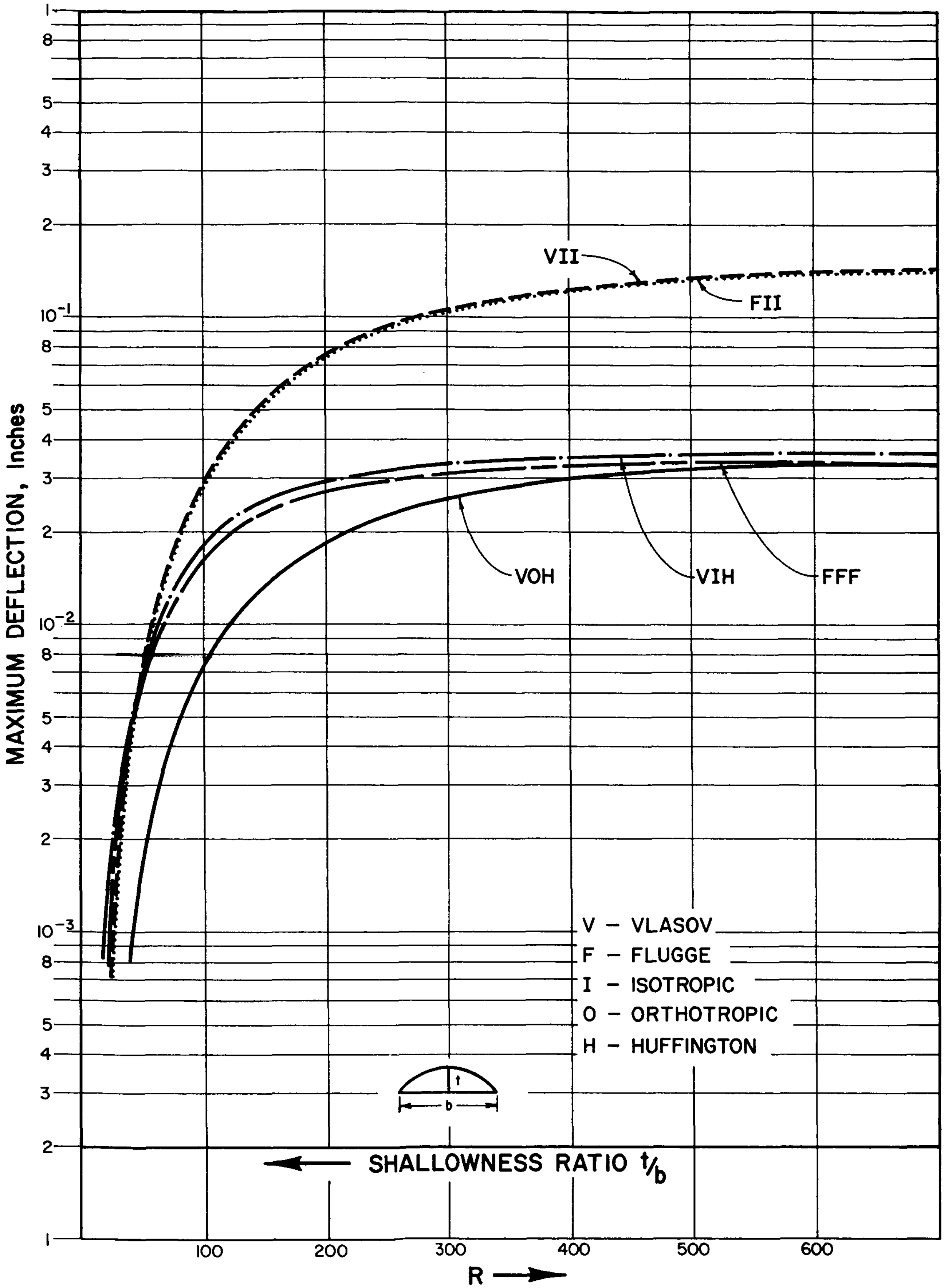


COMPLETE SHELL



DETAILS OF RIB-STIFFENED SHALLOW CYLINDRICAL PANEL

FIGURE I



MAXIMUM DEFLECTIONS VS. SHALLOWNESS RATIOS FOR THE THEORIES THAT WERE INVESTIGATED

FIGURE 2

References

1. Bennett, V. O., "The Analysis of Orthogonally Stiffened Plates" M. S. Thesis, Texas A&M University, (May 1966).

2. Dow, N., Libove, C., Hubka, R., "Formulas for the Elastic Constants of Plates With Integral Waffle-like Stiffening," NASA Report No. 1195, Washington, D. C. (May 1953).

3. Witt, R., Hoppmann, W., II, Buxbaum, R.S. "Determination of Elastic Constants of Orthotropic Materials With Special Reference to Laminates," American Society for Testing Materials, Bulletin No. 194, (Dec. 1953).

4. Hoppmann, W., II, "Bending of Orthogonally Stiffened Plates," Journal of Applied Mechanics, Vol. 22, No. 2 (June 1955).

5. Huffington, N., J., "Theoretical Determination of Rigidity Properties of Orthogonally Stiffened Plates," Journal of Applied Mechanics, Vol. 23, No. 1 (March 1956).

6. Huffington, N., Jr., Schumacker, R., Irwin, R., "Bending Athwart a Paralled Stiffened Plate," Martin Research Report RR-53 (December 1964).

7. Huffington, N., Jr., Schumacker, R., "Flexure of Parallel Stiffened Plates," Martin Research Report RR-59 (January 1965).

8. Tidwell, D. R., "Analysis of Structurally and Intrinsically Orthotropic Shells of Revolution by the Matrix Displacement Method" Doctoral Dissertation, Texas A&M University, (Aug. 1966).

9. Wheeler, O. E., "Transverse Shear Deformation in Thin Shells of Revolution", Doctoral Dissertation, Texas A&M University (May 1966).

10. Chaput, A. J., "A General Displacement Analysis of an Orthotropic Shell of Revolution", Doctoral Dissertation, Texas A&M University (May 1966).

11. Sanders, J. L., Jr., "An Improved First-Approximation Theory for Thin Shells," National Aeronautics and Space Administration Technical Report R-24 (1959).

12. Hunter, D. T., "A Displacement Analysif for Symmetrically Loaded Plane Anisotropic Shells of Revolution, Doctoral Dissertation, Texas A&M University, (Jan. 1967).

13. Trainer, L. D., "The Analysis of Rib Stiffened Shallow Shells by Means of the Compliance Method", M. S. Thesis, Texas A&M University, (Jan. 1967).